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Derivation of arithmetical functions under the Dirichlet convolution

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Abstract

We present the group-theoretic structure of the classes of multiplicative and firmly multiplicative arithmetical functions of several variables under the Dirichlet convolution, and we give characterizations of these two classes in terms of a derivation of arithmetical functions.

Keywords: arithmetical function of several variables, multiplicative function, firmly multiplicative function, derivation, Dirichlet convolution

MSC: 11A25

1 Introduction

Let R be an integral domain. Let $A_r(R) := \{f : R^r \to \mathbb{C}\}$ denote the set of all arithmetical functions of r variables. An arithmetical function $f \in A_r(R)$ is said to be multiplicative if $f(1, \ldots, 1)$ is invertible and

$$f(m_1 n_1, \ldots, m_r n_r) = f(m_1, \ldots, m_r) f(n_1, \ldots, n_r)$$

for all positive integers $m_1, \ldots, m_r$ and $n_1, \ldots, n_r$ with $(m_1 \cdots m_r, n_1 \cdots n_r) = 1$. Clearly, $f(1, \ldots, 1) = 1$ if f is multiplicative. Further, a multiplicative function is completely determined by its values at $(p^{s_1}, \ldots, p^{s_r})$, where p runs through all primes and $s_1, \ldots, s_r \geq 0$. This concept of a multiplicative function coincides with that presented in [9, 12] but differs from that used in [1].

We say that an arithmetical function $f \in A_r(R)$ is *firmly multiplicative* if $f(1, \ldots, 1)$ is invertible and

$$f(m_1 n_1, \ldots, m_r n_r) = f(m_1, \ldots, m_r) f(n_1, \ldots, n_r)$$

for all positive integers $m_1, \ldots, m_r$ and $n_1, \ldots, n_r$ with $(m_1, n_1) = \cdots = (m_r, n_r) = 1$. Clearly, $f(1, \ldots, 1) = 1$ if f is firmly multiplicative. Further, a firmly multiplicative function is completely determined by its values at $(1, \ldots, 1, p^s, 1, \ldots, 1)$, where p runs through all primes and $s \geq 1$, i.e., it is completely determined by its values at $(n_1, \ldots, n_r)$, where one of $n_1, \ldots, n_r$ is a prime power (> 1) and the others are $= 1$. The concept of a firmly multiplicative function coincides with the concept of a multiplicative function presented in [1].

For example, if $f_1, \ldots, f_r$ are multiplicative functions of one variable, then the arithmetical function f of r variables defined as $f(n_1, \ldots, n_r) = f(n_1) \cdots f(n_r)$ is firmly multiplicative. On the other hand, the function $\gcd(n_1, \ldots, n_r)$ is multiplicative but not firmly multiplicative for $r \geq 2$. Further examples can be found, e.g., in [5, 10, 11]. A survey on multiplicative arithmetical functions of several variables is presented in [11].

An arithmetical function $f \in A_r(R)$ is said to be additive if

$$f(m_1 n_1, \ldots, m_r n_r) = f(m_1, \ldots, m_r) + f(n_1, \ldots, n_r)$$

for all positive integers $m_1, \ldots, m_r$ and $n_1, \ldots, n_r$ with $(m_1 \cdots m_r, n_1 \cdots n_r) = 1$, and an arithmetical function $f \in A_r(R)$ is said to be completely additive if

$$f(m_1 n_1, \ldots, m_r n_r) = f(m_1, \ldots, m_r) + f(n_1, \ldots, n_r)$$

for all positive integers $m_1, \ldots, m_r$ and $n_1, \ldots, n_r$. We say that an arithmetical function $f \in A_r(R)$ is *firmly additive* if

$$f(m_1 n_1, \ldots, m_r n_r) = f(m_1, \ldots, m_r) + f(n_1, \ldots, n_r)$$

for all positive integers $m_1, \ldots, m_r$ and $n_1, \ldots, n_r$ with $(m_1, n_1) = \cdots = (m_r, n_r) = 1$. Each completely additive function is firmly additive, and each firmly additive function is additive. Clearly, $f(1, \ldots, 1) = 0$ if f is additive.

The Dirichlet convolution of $f, g \in A_r(R)$ is defined as

$$(f * g)(n_1, \ldots, n_r) = \sum_{d_1 | n_1} \cdots \sum_{d_r | n_r} f(d_1, \ldots, d_r) g(n_1/d_1, \ldots, n_r/d_r).$$

Let $\delta \in A_1(R)$ be defined as $\delta(1) = 1$ and $\delta(n) = 0$ otherwise, and let $\delta \in A_r(R)$ be defined as $\delta(n_1, \ldots, n_r) = \delta(n_1) \cdots \delta(n_r)$. Then δ is the identity under the Dirichlet convolution, and it is firmly multiplicative. The Dirichlet inverse f^{-1} of $f \in A_r(R)$ exists if and only if $f(1, \ldots, 1)$ is invertible.

Derivations for arithmetical functions have been presented, e.g., in [1, 2, 4, 3, 6]. A certain property of multiplicative type functions in terms of derivations is well known [1, 2, 4], see also [7, 8]. In this paper we adopt the derivation given in [1] and utilize the method of Rearick [7] to obtain the above mentioned derivation-related property for multiplicative and firmly multiplicative functions, see Theorems 3 and 4. We obtain a short proof for this property of firmly multiplicative functions given in [1, Theorem 5]. We also show that this property of firmly multiplicative functions actually holds only for firmly multiplicative functions and therefore is, in fact, a characterization of firmly multiplicative functions. An analogous characterization of multiplicative functions is also presented.

We begin by showing the group-theoretic structure of multiplicative and firmly multiplicative functions under the Dirichlet convolution.

2 Group-theoretic structure

Theorem 1. (a) *The set of all arithmetical functions* $f \in A_r(R)$ *with* $f(1, \ldots, 1)$ *invertible is an abelian group under the Dirichlet convolution.*

(b) *The set of all multiplicative functions forms a subgroup of the abelian group in (a).*

(c) *The set of all firmly multiplicative functions forms a subgroup of the abelian group in (b).*

Proof. We prove only that if f is multiplicative, then f^{-1} is multiplicative. Let f be multiplicative. We proceed by induction on $m_1 \cdots m_r n_1 \cdots n_r$ to prove that

$$f^{-1}(m_1 n_1, \ldots, m_r n_r) = f^{-1}(m_1, \ldots, m_r) f^{-1}(n_1, \ldots, n_r) \tag{1}$$

whenever $(m_1 \cdots m_r, n_1 \cdots n_r) = 1$.

Assume that $m_1 \cdots m_r n_1 \cdots n_r = 1$. Then $m_1 = \cdots = m_r = n_1 = \cdots = n_r = 1$, and since f is multiplicative, $f(1, \ldots, 1) = 1$. This implies that $f^{-1}(1, \ldots, 1) = 1$, and therefore both sides of (1) equal 1 and thus (1) holds.

Assume that $m_1 \cdots m_r n_1 \cdots n_r \neq 1$, $(m_1 \cdots m_r, n_1 \cdots n_r) = 1$ and

$$f^{-1}(m'_1 n'_1, \ldots, m'_r n'_r) = f^{-1}(m'_1, \ldots, m'_r) f^{-1}(n'_1, \ldots, n'_r) \tag{2}$$

whenever $(m'_1 \cdots m'_r, n'_1 \cdots n'_r) = 1$ and $m'_1 \cdots m'_r n'_1 \cdots n'_r < m_1 \cdots m_r n_1 \cdots n_r$. We show that (1) holds. If $m_1 \cdots m_r = 1$ or $n_1 \cdots n_r = 1$, then (1) holds. Assume that $m_1 \cdots m_r \neq 1$ and $n_1 \cdots n_r \neq 1$. Now, $(f * f^{-1})(m_1 n_1, \ldots, m_r n_r) = 0$, that is,

$$f^{-1}(m_1 n_1, \ldots, m_r n_r) = - \sum_{\substack{d_1 | m_1 n_1, \ldots, d_r | m_r n_r \\ d_1 \cdots d_r > 1}} f(d_1, \ldots, d_r) f^{-1}(m_1 n_1 / d_1, \ldots, m_r n_r / d_r).$$

Since $(m_1 \cdots m_r, n_1 \cdots n_r) = 1$, the above summation can be written as

$$-\sum_{\substack{a_1|m_1,\ldots,a_r|m_r \\ b_1|n_1,\ldots,b_r|n_r \\ a_1\cdots a_r b_1\cdots b_r>1}} f(a_1b_1,\ldots,a_rb_r)f^{-1}((m_1/a_1)(n_1/b_1),\ldots,(m_r/a_r)(n_r/b_r)).$$

On the basis of multiplicativity of f and equation (2) this becomes

$$-\sum_{\substack{a_1|m_1,\ldots,a_r|m_r \\ b_1|n_1,\ldots,b_r|n_r \\ a_1\cdots a_r b_1\cdots b_r>1}} f(a_1,\ldots,a_r)f(b_1,\ldots,b_r)f^{-1}(m_1/a_1,\ldots,m_r/a_r)f^{-1}(n_1/b_1,\ldots,n_r/b_r).$$

Arranging the terms we obtain

$$\begin{aligned}
&f^{-1}(m_1n_1,\ldots,m_rn_r) \\
&= -f^{-1}(m_1,\ldots,m_r)\sum_{\substack{b_1|n_1,\ldots,b_r|n_r \\ b_1\cdots b_r>1}} f(b_1,\ldots,b_r)f^{-1}(n_1/b_1,\ldots,n_r/b_r) \\
&\quad - f^{-1}(n_1,\ldots,n_r)\sum_{\substack{a_1|m_1,\ldots,a_r|m_r \\ a_1\cdots a_r>1}} f(a_1,\ldots,a_r)f^{-1}(m_1/a_1,\ldots,m_r/a_r) \\
&\quad - \sum_{\substack{a_1|m_1,\ldots,a_r|m_r \\ a_1\cdots a_r>1}} f(a_1,\ldots,a_r)f^{-1}(m_1/a_1,\ldots,m_r/a_r) \\
&\qquad \times \sum_{\substack{b_1|n_1,\ldots,b_r|n_r \\ b_1\cdots b_r>1}} f(b_1,\ldots,b_r)f^{-1}(n_1/b_1,\ldots,n_r/b_r) \\
&= f^{-1}(m_1,\ldots,m_r)f^{-1}(n_1,\ldots,n_r) + f^{-1}(m_1,\ldots,m_r)f^{-1}(n_1,\ldots,n_r) \\
&\quad - f^{-1}(m_1,\ldots,m_r)f^{-1}(n_1,\ldots,n_r) \\
&= f^{-1}(m_1,\ldots,m_r)f^{-1}(n_1,\ldots,n_r).
\end{aligned}$$

Thus (1) holds. This shows that f^{-1} is multiplicative.

3 Derivation

Let $\psi \in A_r(R)$ be a completely additive function. We define the derivation $D_\psi \colon A_r(R) \to A_r(R)$ by

$$D_\psi(f)(n_1,\ldots,n_r) = f(n_1,\ldots,n_r)\psi(n_1,\ldots,n_r).$$

In what follows we write $\mathbf{n} = (n_1, \ldots, n_r)$ for the sake of brevity.

We first present the basic properties of the derivation.

Theorem 2. *For any $f, g \in A_r(R)$ and $c \in R$,*

(a) $D_\psi(f+g) = D_\psi(f) + D_\psi(g)$,

(b) $D_\psi(f*g) = f*D_\psi(g) + g*D_\psi(f)$,

(c) $D_\psi(cf) = cD_\psi(f)$.

Proof. See [1].

We next present the promised characterizations of multiplicative and firmly multiplicative functions in terms of the derivation.

Theorem 3. *Let $f \in A_r(R)$ be an arithmetical function with $f(1, \ldots, 1) = 1$. If f is multiplicative, then $(D_\psi(f)*f^{-1})(\mathbf{n}) = 0$ whenever $\mathbf{n}$ is not of the form $(p^{s_1}, \ldots, p^{s_r})$, where p is a prime number and $s_1, \ldots, s_r \geq 0$. The converse holds provided that $\psi(\mathbf{n}) \neq 0$ for all $\mathbf{n} \neq (1, \ldots, 1)$.*

Proof. Assume that f is multiplicative. Assume also that $\mathbf{n}$ is not of the form $(p^{s_1}, \ldots, p^{s_r})$, where p is a prime number and $s_1, \ldots, s_r \geq 0$. Then $\mathbf{n}$ can be written as $\mathbf{n} = (k_1 m_1, \ldots, k_r m_r)$, where $(k_1 \cdots k_r, m_1 \cdots m_r) = 1$ with $k_1 \cdots k_r > 1$ and $m_1 \cdots m_r > 1$. We show that $(D_\psi(f)*f^{-1})(\mathbf{n}) = 0$. Since f is multiplicative, we have

$$\begin{aligned}
&(D_\psi(f)*f^{-1})(\mathbf{n}) \\
&= \sum_{d_1 | k_1 m_1} \cdots \sum_{d_r | k_r m_r} f(d_1, \ldots, d_r) f^{-1}(k_1 m_1/d_1, \ldots, k_r m_r/d_r)\psi(d_1, \ldots, d_r) \\
&= \sum_{\substack{a_1 | k_1 \\ b_1 | m_1}} \cdots \sum_{\substack{a_r | k_r \\ b_r | m_r}} f(a_1, \ldots, a_r) f(b_1, \ldots, b_r) f^{-1}(k_1/a_1, \ldots, k_r/a_r) \\
&\qquad \times f^{-1}(m_1/b_1, \ldots, m_r/b_r)\Big(\psi(a_1, \ldots, a_r) + \psi(b_1, \ldots, b_r)\Big).
\end{aligned}$$

Rearranging the terms we obtain

$$
\begin{aligned}
&(D_\psi(f) * f^{-1})(\mathbf{n}) \\
&= \sum_{a_1|k_1} \cdots \sum_{a_r|k_r} f(a_1, \ldots, a_r) f^{-1}(k_1/a_1, \ldots, k_r/a_r) \psi(a_1, \ldots, a_r) \\
&\quad \times \sum_{b_1|m_1} \cdots \sum_{b_r|m_r} f(b_1, \ldots, b_r) f^{-1}(m_1/b_1, \ldots, m_r/b_r) \\
&\quad + \sum_{b_1|m_1} \cdots \sum_{b_r|m_r} f(b_1, \ldots, b_r) f^{-1}(m_1/b_1, \ldots, m_r/b_r) \psi(b_1, \ldots, b_r) \\
&\quad \times \sum_{a_1|k_1} \cdots \sum_{a_r|k_r} f(a_1, \ldots, a_r) f^{-1}(k_1/a_1, \ldots, k_r/a_r) \\
&= (D_\psi(f) * f^{-1})(\mathbf{k}) \delta(\mathbf{m}) + (D_\psi(f) * f^{-1})(\mathbf{m}) \delta(\mathbf{k}).
\end{aligned}
$$

Since $\mathbf{k} \neq (1, \ldots, 1)$ and $\mathbf{m} \neq (1, \ldots, 1)$, we have $\delta(\mathbf{k}) = \delta(\mathbf{m}) = 0$, and therefore $(D_\psi(f) * f^{-1})(\mathbf{n}) = 0$.

Conversely, assume that $(D_\psi(f) * f^{-1})(\mathbf{n}) = 0$ whenever $\mathbf{n}$ is not of the form $(p^{s_1}, \ldots, p^{s_r})$, where p is a prime number and $s_1, \ldots, s_r \geq 0$. We show that f is multiplicative. Let $g \in A_r(R)$ be the arithmetical function defined as $g(1, \ldots, 1) = 1$ and

$$g(\mathbf{n}) = \prod_p f(p^{n_1(p)}, \ldots, p^{n_r(p)}),$$

where $n_i = \prod_p p^{n_i(p)}$ is the canonical factorization of n_i for $i = 1, \ldots, r$. Then g is multiplicative. We show that $f = g$.

We first show that $D_\psi(f) * f^{-1} = D_\psi(g) * g^{-1}$. Clearly, $f(\mathbf{n}) = g(\mathbf{n})$ if $\mathbf{n}$ is of the form $(p^{s_1}, \ldots, p^{s_r})$, where p is a prime number and $s_1, \ldots, s_r \geq 0$. This implies that $D_\psi(f)(\mathbf{n}) = D_\psi(g)(\mathbf{n})$ and $f^{-1}(\mathbf{n}) = g^{-1}(\mathbf{n})$ if $\mathbf{n}$ is of the form $(p^{s_1}, \ldots, p^{s_r})$, where p is a prime number and $s_1, \ldots, s_r \geq 0$. Thus $(D_\psi(f) * f^{-1})(\mathbf{n}) = (D_\psi(g) * g^{-1})(\mathbf{n})$ if $\mathbf{n}$ is of the form $(p^{s_1}, \ldots, p^{s_r})$, where p is a prime number and $s_1, \ldots, s_r \geq 0$. Since g is multiplicative, on the basis of the first part of this theorem $(D_\psi(g) * g^{-1})(\mathbf{n}) = 0$ if $\mathbf{n}$ is not of the form $(p^{s_1}, \ldots, p^{s_r})$, where p is a prime number and $s_1, \ldots, s_r \geq 0$. Thus, by the assumption on f, $(D_\psi(f) * f^{-1})(\mathbf{n}) = (D_\psi(g) * g^{-1})(\mathbf{n})$ for all $\mathbf{n}$.

We now show that $f(\mathbf{n}) = g(\mathbf{n})$ for all $\mathbf{n} = (n_1, \ldots, n_r)$ applying induction on $\Omega(n_1) + \cdots + \Omega(n_r)$, where $\Omega(n_i)$ is the total number of prime divisors of n_i each counted according to its multiplicity with $\Omega(1) = 0$. If $\Omega(n_1) + \cdots + \Omega(n_r) = 0$, then $\mathbf{n} = (1, \ldots, 1)$ and $f(\mathbf{n}) = g(\mathbf{n}) = 1$. Assume that $f(\mathbf{n}) = g(\mathbf{n})$ for $\Omega(n_1) + \cdots + \Omega(n_r) < k$. Then $f^{-1}(\mathbf{n}) = g^{-1}(\mathbf{n})$ for $\Omega(n_1) + \cdots + \Omega(n_r) < k$. Let $\mathbf{m}$ be such that $\Omega(m_1) + \cdots + \Omega(m_r) = k$. We show that $f(\mathbf{m}) = g(\mathbf{m})$. We have shown that $(D_\psi(f) * f^{-1})(\mathbf{m}) = (D_\psi(g) * g^{-1})(\mathbf{m})$, which means that

$$\sum_{d_1|m_1,\ldots,d_r|m_r} f(d_1, \ldots, d_r) f^{-1}(m_1/d_1, \ldots, m_r/d_r) \psi(d_1, \ldots, d_r)$$

$$= \sum_{d_1|m_1,\ldots,d_r|m_r} g(d_1, \ldots, d_r) g^{-1}(m_1/d_1, \ldots, m_r/d_r) \psi(d_1, \ldots, d_r).$$

Since $\psi(1, \ldots, 1) = 0$, we have

$$f(\mathbf{m})\psi(\mathbf{m}) + \sum_{\substack{d_1|m_1,\ldots,d_r|m_r \\ (d_1,\ldots,d_r)\neq(1,\ldots,1) \\ (d_1,\ldots,d_r)\neq(m_1,\ldots,m_r)}} f(d_1, \ldots, d_r) f^{-1}(m_1/d_1, \ldots, m_r/d_r) \psi(d_1, \ldots, d_r)$$

$$= g(\mathbf{m})\psi(\mathbf{m}) + \sum_{\substack{d_1|m_1,\ldots,d_r|m_r \\ (d_1,\ldots,d_r)\neq(1,\ldots,1) \\ (d_1,\ldots,d_r)\neq(m_1,\ldots,m_r)}} g(d_1, \ldots, d_r) g^{-1}(m_1/d_1, \ldots, m_r/d_r) \psi(d_1, \ldots, d_r).$$

Since $f(\mathbf{n}) = g(\mathbf{n})$ and $f^{-1}(\mathbf{n}) = g^{-1}(\mathbf{n})$ for $\Omega(n_1) + \cdots + \Omega(n_r) < k$, the summations above are equal. Thus $f(\mathbf{m})\psi(\mathbf{m}) = g(\mathbf{m})\psi(\mathbf{m})$, and since $\psi(\mathbf{m}) \neq 0$, we have $f(\mathbf{m}) = g(\mathbf{m})$. We have thus shown by induction that $f = g$. Since g is multiplicative, f is multiplicative. This completes the proof.

Theorem 4. *Let $f \in A_r(R)$ be an arithmetical function with $f(1, \ldots, 1) = 1$. If f is firmly multiplicative, then $(D_\psi(f) * f^{-1})(\mathbf{n}) = 0$ whenever $\mathbf{n} = (n_1, \ldots, n_r)$ is not of the form $\mathbf{n} = (1, \ldots, 1, p^s, 1, \ldots, 1)$, where p is a prime number and $s \geq 0$ (i.e. $(D_\psi(f) * f^{-1})(\mathbf{n}) = 0$ unless one of $n_1, \ldots, n_r$ is a prime power (≥ 1) and the others are $= 1$). The converse holds provided that $\psi(\mathbf{n}) \neq 0$ for all $\mathbf{n} \neq (1, \ldots, 1)$.*

Proof. Assume that f is firmly multiplicative. Assume also that $\mathbf{n}$ is not of the form $(1, \dots, 1, p^s, 1, \dots, 1)$, where p is a prime number and $s \geq 0$. Then $\mathbf{n}$ can be written as $\mathbf{n} = (k_1 m_1, \dots, k_r m_r)$, where $(k_1, m_1) = \cdots = (k_r, m_r) = 1$ with $k_1 \cdots k_r > 1$ and $m_1 \cdots m_r > 1$. We show that $(D_\psi(f) * f^{-1})(\mathbf{n}) = 0$. Since f is firmly multiplicative, we have

$$\begin{aligned}
&(D_\psi(f) * f^{-1})(\mathbf{n}) \\
&= \sum_{d_1 | k_1 m_1} \cdots \sum_{d_r | k_r m_r} f(d_1, \dots, d_r) f^{-1}(k_1 m_1/d_1, \dots, k_r m_r/d_r) \psi(d_1, \dots, d_r) \\
&= \sum_{\substack{a_1 | k_1 \\ b_1 | m_1}} \cdots \sum_{\substack{a_r | k_r \\ b_r | m_r}} f(a_1, \dots, a_r) f(b_1, \dots, b_r) f^{-1}(k_1/a_1, \dots, k_r/a_r) \\
&\qquad \times f^{-1}(m_1/b_1, \dots, m_r/b_r) \Big(\psi(a_1, \dots, a_r) + \psi(b_1, \dots, b_r) \Big) \\
&= (D_\psi(f) * f^{-1})(\mathbf{k}) \delta(\mathbf{m}) + (D_\psi(f) * f^{-1})(\mathbf{m}) \delta(\mathbf{k}).
\end{aligned}$$

Since $\mathbf{k} \neq (1, \dots, 1)$ and $\mathbf{m} \neq (1, \dots, 1)$, we have $\delta(\mathbf{k}) = \delta(\mathbf{m}) = 0$, and therefore $(D_\psi(f) * f^{-1})(\mathbf{n}) = 0$.

Conversely, assume that $(D_\psi(f) * f^{-1})(\mathbf{n}) = 0$ whenever $\mathbf{n}$ is not of the form $(1, \dots, 1, p^s, 1, \dots, 1)$, where p is a prime number and $s \geq 0$. We show that f is firmly multiplicative. Let $g \in A_r(R)$ be the arithmetical function defined as $g(1, \dots, 1) = 1$ and

$$g(\mathbf{n}) = \prod_p \Big(f(p^{n_1(p)}, 1, \dots, 1) f(1, p^{n_2(p)}, 1, \dots, 1) \cdots f(1, \dots, 1, p^{n_r(p)}) \Big),$$

where $n_i = \prod_p p^{n_i(p)}$ is the canonical factorization of n_i for $i = 1, 2, \dots, r$. Then g is firmly multiplicative. We show that $f = g$.

We first show that $D_\psi(f) * f^{-1} = D_\psi(g) * g^{-1}$. Clearly, $f(\mathbf{n}) = g(\mathbf{n})$ if $\mathbf{n}$ is of the form $(1, \dots, 1, p^s, 1, \dots, 1)$, where p is a prime number and $s \geq 0$. This implies that $D_\psi(f)(\mathbf{n}) = D_\psi(g)(\mathbf{n})$ and $f^{-1}(\mathbf{n}) = g^{-1}(\mathbf{n})$ if $\mathbf{n}$ is of the form $(1, \dots, 1, p^s, 1, \dots, 1)$, where p is a prime number and $s \geq 0$. Thus $(D_\psi(f) * f^{-1})(\mathbf{n}) = (D_\psi(g) * g^{-1})(\mathbf{n})$ if $\mathbf{n}$ is of the form $(1, \dots, 1, p^s, 1, \dots, 1)$, where p is a prime number and $s \geq 0$. Since g is firmly multiplicative, on the basis of the first part of this theorem $(D_\psi(g) * g^{-1})(\mathbf{n}) = 0$ if $\mathbf{n}$ is not of the form $(1, \ldots, 1, p^s, 1, \ldots, 1)$, where p is a prime number and $s \geq 0$. Thus, by the assumption on f, $(D_\psi(f) * f^{-1})(\mathbf{n}) = (D_\psi(g) * g^{-1})(\mathbf{n})$ for all $\mathbf{n}$.

We can show that $f(\mathbf{n}) = g(\mathbf{n})$ for all $\mathbf{n} = (n_1, \ldots, n_r)$ applying induction on $\Omega(n_1) + \cdots + \Omega(n_r)$ exactly in the same way as in the proof of Theorem 3. Thus $f = g$, and since g is firmly multiplicative, f is firmly multiplicative. This completes the proof.

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